MATH 8 UNIT 1 part 1: Systems of Linear Equations and Matrices

Review from Math 3: 10.1 and 10.2 Linear Systems of Equations

Linear System in two variables: ____

Solution is an ____

Warm up: Solve the following 2X2 Linear Systems (2 equations with 2 unknowns):

$\begin{cases} 2x+y=-1 \\ -4x+6y=42 \end{cases}$ $\begin{cases} x-3y=5 \\ -2x+6y=4 \end{cases}$ $\begin{cases} 2x+y=4 \\ -6x-3y=-12 \end{cases}$

Case: ____________________ ______________________ ___________________

Methods(thus far): 1)______________________2)_____________________3)______________________

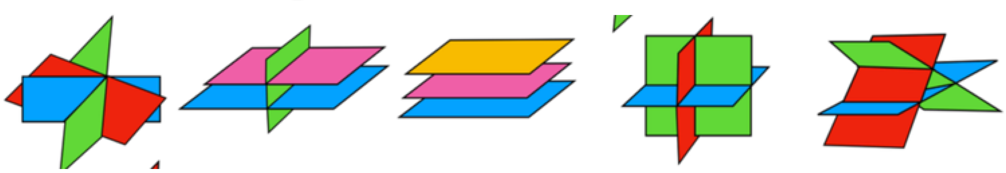
Linear Systems in 3 Variables

3 variables => 3 dimensions

Solutions are ________________________

Graph of a linear equation in 3 variables is a ____________________

Many cases for solutions to a linear system in three variables:



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____________ ______________ ________________ ________________ _______________

Methods (thus far) 1)_________________ 2)________________________

Example :

$$\begin{cases} x + 2y - 2z = -2 \\ -5x - 9y + 4z = 3 \\ 3x + 4y - 5z = -3 \end{cases}$$

Special case 1 example:

$$\begin{cases} 2x + y - z = -2 \\ x + 2y - z = -9 \\ x - 4y + z = 1 \end{cases}$$

Special case 2 example:

$$\begin{cases} x - 2y - z = 8 \\ 2x - 3y + z = 23 \\ 4x - 5y + 5z = 53 \end{cases}$$

Writing the solution to a dependent system.

10.3 Introduction to Matrices , Gaussian Elimination, Gauss-Jordon Method

Matrix:

Size:

Square matrix:

Subscript Notation: Let a_{ij} be the entry of matrix A in row i and column j. If A is an mxn matrix, then

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots\cdots & a_{1n} \\ a & a & a & a & \cdots\cdots & a \\ \vdots & & & & & \\ a & a & a & a & \cdots\cdots & a \end{bmatrix}$$

So for matrix A= $\begin{bmatrix} 3 & 4 & 9 \\ -1 & 8 & 2 \\ 0 & -2 & 5 \end{bmatrix}$,

$a_{12} =$ __________

$a_{23} =$ __________

________ $= 0$

The Augmented Matrix of a Linear System

A system of linear equations can be represented by a matrix called an augmented matrix.

EX: System: $\begin{cases} 2x + y - z = -2 \\ x + 2y - z = -9 \\ x - 4y + z = 1 \end{cases}$ => Augmented Matrix

EX: Augmented Matrix: $\left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ 7 & 9 & 2 & 1 \\ 4 & 1 & -5 & 5 \end{array}\right]$ => System

EX: Write the following special augmented matrices as a system, then solve the system:

a) Row Echelon Form: $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & -3 \end{array}\right]$

b) Reduced Row Echelon Form $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$

Observation: If an augmented matrix is in Row Echelon Form, or Reduced Row Echelon form, it is easy to solve the corresponding system!

What is Row/Reduce Row Echelon Form?

Row Echelon Form: $\left[\begin{array}{ccc|c} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 1 & x \end{array}\right]$

1. The first nonzero number in each row is 1. This is called the leading entry.
2. Any leading 1 is below and to the right of a previous leading 1.
3. Any all-zero rows are placed at the bottom on the matrix.

Reduced Row Echelon Form $\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & x \end{array}\right]$

Satisfies the above conditions, and

4. Every number above and below each leading entry is a 0.

EX: Are the following in Row Echelon Form, Reduced Row Echelon Form or Neither?

$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 1 & -3 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & 0 & -3 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Gaussian/Gauss Jordan Methods

Goal of Gaussian Elimination: Given a linear system of equations, perform a series of "allowed row operations" to an augmented matrix to find a matrix in row echelon form representing an equivalent linear system. Then solve the simpler system. (If the process is continued to obtain reduced row echelon form, this is called Gauss-Jordan method.)

Illustration of the method:

Solve:

$$\begin{cases} x+2y-2z=-2 & (1) \\ -5x-9y+4z=3 & (2) \\ 3x+4y-5z=-3 & (3) \end{cases} \text{ Eliminate x } \begin{cases} \begin{Bmatrix} 5Eq(1)\to 5x+10y-10z=-10 \\ Eq.(2)\to -5x-9y+4z=3 \end{Bmatrix} \to ADD \to \quad y-6z=-7 \quad (4) \\ \begin{Bmatrix} -3Eq(1)\to -3x-6y+6z=6 \\ Eq.(3)\to \quad 3x+4y-5z=-3 \end{Bmatrix} \to ADD \to \quad -2y+z=3 \quad (5) \end{cases}$$

$$\text{Elim. Y: } \begin{cases} 2Eq(4)\to 2y-12z=-14 \\ Eq.(5)\to \quad -2y+z=3 \end{cases} \quad ADD \to -11z=-11 \quad so \quad z=1$$

Augmented matrix, Row operations, Row Echelon Form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ -5 & -9 & 4 & 3 \\ 3 & 4 & -5 & -3 \end{array}\right] \cdots \begin{cases} 5R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -6 & -7 \\ 0 & -2 & 1 & 3 \end{array}\right] \cdots \{ 2R_2 + R_3 \to R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -6 & -7 \\ 0 & 0 & -11 & -11 \end{array}\right] \cdots \{ -\tfrac{1}{11}R_3 \to R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -6 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Now write the corresponding system and use back substitution to solve.:

Allowed Elementary Row Operations:

1. ____________________
2. _____________________
3. _____________________

EX: Practicing Random Row Operations:

$$\left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ 7 & 9 & 2 & 1 \\ 4 & 1 & -5 & 5 \end{array}\right] \Rightarrow \quad -3R_2 \to R_2 \Rightarrow \left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ & & & \\ 4 & 1 & -5 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ 7 & 9 & 2 & 1 \\ 4 & 1 & -5 & 5 \end{array}\right] \quad R_1 \leftrightarrow R_2 \Rightarrow \left[\begin{array}{ccc|c} & & & \\ & & & \\ 4 & 1 & -5 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ 7 & 9 & 2 & 1 \\ 4 & 1 & -5 & 5 \end{array}\right] \quad \Rightarrow 5R_3 + R_2 \to R_2 \Rightarrow \left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ & & & \\ 4 & 1 & -5 & 5 \end{array}\right]$$

The key to Gaussian elimination is to learn how to choose row operations that will yield row echelon form.

EX: Solve: $\begin{cases} 3x - y + 5z = 14 \\ x + 2y - 2z = 10 \\ x - y + 3z = 4 \end{cases}$

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 14 \\ 1 & 2 & -2 & 10 \\ 1 & -1 & 3 & 4 \end{array}\right] \text{------------}\rightarrow$$

EX: Solve: $\begin{cases} 3x + y - z = \frac{2}{3} \\ 2x - y + z = 1 \\ 4x + 2y = \frac{8}{3} \end{cases}$

First write the augmented matrix, then obtain a 1 in position a_{11}, and then use that 1 to get zeros below it.

EX: 4X4 Gaussian Elimination / Gauss Jordan Example

Solve: $\begin{cases} x + \quad z + 2w = 6 \\ y - 2z \quad = -3 \\ x + 2y - z \quad = -2 \\ 2x + y + 3z - 2w = 0 \end{cases}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 1 & 2 & -1 & 0 & -2 \\ 2 & 1 & 3 & -2 & 0 \end{bmatrix} \xrightarrow[-2R_1+R_4 \to R_4]{-R_1+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 2 & -2 & -2 & -8 \\ 0 & 1 & 1 & -6 & -12 \end{bmatrix} \xrightarrow[-R_2+R_4 \to R_4]{-2R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 2 & -2 & -2 \\ 0 & 0 & 3 & -6 & -9 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & -6 & -9 \end{bmatrix} \xrightarrow{-3R_3+R_4 \to R_4} \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -3 & -6 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_4 \to R_4} \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This is row echelon form. If using Gaussian elimination you can stop your row operations here, write the corresponding system, and use back substitution to find the solution. If using Gauss-Jordan then continue with row operations until reduced row echelon form is achieved.

Continuing, getting zeros above the leading ones…

$$\xrightarrow[-2R_4+R_1 \to R_1]{R_4+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[-R_3+R_1 \to R_1]{2R_3+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

From here we can see the solution, x=1, y=-1, z=1, w=2, that is (1, -1, 1, 2).

There are many other sequences of row operations that are acceptable, but they must achieve the same solution in the end. With practice, you will be able to combine more operations into each step.

Gaussian Elimination: Dependent and Inconsistent Case Examples

Dependent System Example:

$$\begin{cases} 6x - y - z = 4 \\ -12x + 2y + 2z = -8 \\ 5x + y - z = 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

Inconsistent System Example: See Book

$$\rightarrow \left[\begin{array}{ccc|c} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 0 & nonzero \end{array}\right]$$

10.4 Algebra of Matrices

Matrix Equality

Two matrices are considered equal if they are the ________________________ and have ___________ corresponding entries.

Matrix Addition and Subtraction

ADDING AND SUBTRACTING MATRICES

Given matrices A and B of like dimensions, addition and subtraction of A and B will produce matrix C or matrix D of the same dimension.

$$A + B = C \text{ such that } a_{ij} + b_{ij} = c_{ij}$$

$$A - B = D \text{ such that } a_{ij} - b_{ij} = d_{ij}$$

Matrix addition is commutative.

$$A + B = B + A$$

It is also associative.

$$(A + B) + C = A + (B + C)$$

Source: Openstax, Algebra and Trigonometr

Scalar Multiplication

SCALAR MULTIPLICATION

Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the scalar multiple cA is

$$cA = c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

Scalar multiplication is distributive. For the matrices A, B, and C with scalars a and b,

$$a(A + B) = aA + aB$$
$$(a + b)A = aA + bA$$

Source: Openstax, Algebra and Trigonometry

Example: If $A=\begin{bmatrix} 4 & 1 \\ -1 & -2 \end{bmatrix}$ $B=\begin{bmatrix} 5 & 7 & -1 \\ 2 & 0 & 3 \\ -3 & 1 & 2 \end{bmatrix}$ $C=\begin{bmatrix} 1 & 7 \\ -2 & -5 \end{bmatrix}$ find

1) A+C

2) A+B

3) Compute 3A

4) Compute 4C-A

Matrix Multiplication

<u>Special case</u>: Row matrix times column matrix.

If R is a $1\times n$ matrix (also called a row vector) $R=\begin{bmatrix} r_1 & r_2 & r_3 & \cdots & r_n \end{bmatrix}$

and C is an $n\times 1$ matrix (also called a column vector) $C=\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$

The their "inner product" RC is the number given by ______________________

General matrix multiplication:

If $A=\left[a_{ij}\right]$ is an $m\times n$matrix and $B=\left[b_{ij}\right]$ is an $n\times k$ matrix, then their product is the $m\times k$ matrix $C=\left[c_{ij}\right]$ where is the inner product of the i th row of A and the j th column of B.

Ex:

$$\begin{bmatrix} 2 & -1 & 7 \\ -3 & 0 & 4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 6 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

Ex: $A=\begin{bmatrix} 4 & 1 \\ -1 & -2 \end{bmatrix}$ $B=\begin{bmatrix} -5 & 0 & 2 \\ 3 & 8 & 1 \end{bmatrix}$ $C=\begin{bmatrix} 1 & 7 \\ -2 & -5 \end{bmatrix}$

Find 1) AC

2) CA

3) A^2

4) AB

4) BA

Notice: Matrix multiplication is NOT ____________________________

<u>(Multiplicative) Identity Matrix</u>

$I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $I_4=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Find: AI_2

(What is the multiplicative identity in real numbers?)

Application of Matrix Multiplication

Suppose the two soccer teams below need to order new equipment. Use matrices to compute the total cost for all the requested equipment.

Equipment Request

	Tigers	Sparks
Goals	2	1
Balls	3	5
Jerseys	12	10

Cost per Item

Goals	100
Balls	12
Jerseys	10

What if there is a lot more data?

Equipment Request

	Galaxy	Sounders	LAFC	NYFC	Tiimbers	Rapids	Bulls	Sporting KC	DC United	SDFC	FC Dallas	Real SL
Goals	22	35	104	18	36	65	8	24	18	84	17	20
Balls	152	100	135	88	217	145	133	188	147	95	140	92
Jerseys	156	50	65	40	88	67	74	132	97	156	187	84
Shorts	60	40	40	35	60	60	60	80	80	110	142	71
Sweatshirts	35	35	38	41	51	63	37	29	34	36	33	29
Cleats	60	40	40	35	60	60	60	80	80	110	142	71
Socks	200	314	350	298	198	275	234	283	222	187	246	222
Shinguards	60	40	40	35	60	60	60	80	80	110	142	71
Gatoratde	578	813	678	598	374	612	833	622	417	123	512	645

10.5 Inverse Matrices

Much like ordinary algebraic equations, we may be asked to solve matrix equations.

Ex: If $A=\begin{bmatrix} 2 & 1 \\ 9 & 5 \end{bmatrix}$, $B=\begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$, $X=\begin{bmatrix} x & y \\ z & w \end{bmatrix}$, solve the matrix equation 3A-2X=B for X

Ex: If $A=\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$, solve the matrix equation AX=B.

(Consider first how you would solve the equation $\frac{2}{3}x=4$ *)*

We seek a matrix A^{-1} such that $A^{-1}A=AA^{-1}=I$. The matrix A^{-1}, if it exists, is called A inverse.

(Note: A^{-1} does not mean $\frac{1}{A}$ here.

How do we find A^{-1}? Consider the following example, which although not how we will actually find inverses, will give us an idea why the method we will learn works.

Ex to motivate inverse process : Find the inverse if A= $\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$

(This example is done on the video linked on the assignment sheet)

<u>Method for finding A^{-1} :</u>

$$\left[\, A \mid I_n \,\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\, I_n \mid A^{-1} \,\right]$$

Using this method on the above matrix:

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$$

Shortcut for finding inverse of a 2x2 matrix:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. If $ad - bc = 0$ then A does not have an inverse.

Example: Given $A=\begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$, find A^{-1}

Check: $A^{-1}A = AA^{-1} = I$

Tip: You can check your answer *as you go* since $A^{-1}A$ should equal I

Solving Systems of Linear Equations as a MATRIX EQUATION

Any linear system can be written in the form AX=B. Then the solution is known to be $X = A^{-1}B$ (provided A^{-1} exists. If A^{-1} does not exist, we have one of the special cases of no solution or infinitely many solutions).)

Ex: Given the system of equations,

$$\begin{cases} 2x - 3y = 1 \\ 3x + 4y = 3 \end{cases}$$

Show that it can be written as a matrix equation AX=B where

If $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Having done this, we find the solution is $X = A^{-1}B$

$$X = A^{-1}B = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

EX: Solve $\begin{cases} x + y = 5 \\ -x + 3y + 4z = 7 \\ 4y + 3z = 4 \end{cases}$ as a matrix equation

10.6 Determinants and Cramer's Rule

Cramer's Rule for solving Linear Systems

We can generate a formula for solving a system of equations by solving the general system:

Eliminate y:

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases} \Rightarrow \begin{cases} \\ \end{cases}$$

So $x = \dfrac{rd - bs}{ad - bc}$

Similarly, if we eliminate x, we get $y = \dfrac{as - cr}{ad - bc}$

Worth remembering? See a pattern? We will return to this formula. But first…

Determinants of 2X2 Matrices

A determinant is a number corresponding to a square matrix, computed by following the processes described below. Determinants have many properties and uses. You will learn more about determinants in Math 10.

2X2 Determinant:

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant of A, denoted det(A) or |A| is computed as follows:

det(A) = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ______________________

Examples:

Back to solving the system $\begin{cases} ax+by=r \\ cx+dy=s \end{cases}$

$$x=\frac{rd-bs}{ad-bc}$$

$$y=\frac{as-cr}{ad-bc}$$

So if D is the determinant of the coefficient matrix: D= $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Dx is like D, but with x's column replaced by the RHS. Dx= $\begin{vmatrix} & \\ & \end{vmatrix}$

Dy is like D, but with with y's column replaced by the RHS. Dy= $\begin{vmatrix} & \\ & \end{vmatrix}$

Then $x=\frac{D_x}{D}$ and $y=\frac{D_y}{D}$ are the solutions to the equation (D not equal zero. If D=0 we have one of the special cases of no solution or infinitely many solutions).

Try it: Solve the system using Cramer's Rule

$$\begin{cases} 2x-3y=1 \\ 3x+4y=3 \end{cases}$$

Cramer's rule is particularly useful when the numbers are complicated.

This method extends to larger nxn linear systems.

$$\begin{cases} 2x + y - z = 3 \\ -x + 2y + 4z = -3 \\ x - 2y - 3z = 4 \end{cases}$$

$$D = \left| \quad \right| \qquad D_x = \left| \quad \right| \qquad D_y = \left| \quad \right| \qquad D_z = \left| \quad \right|$$

$$x = \frac{D_x}{D} = \qquad y = \frac{D_y}{D} = \qquad z = \frac{D_z}{D} =$$

General nxn determinants.

First some terminology:

The minor, M_{ij}, of entry a_{ij} is defined to be the determinant of the matrix remaining when row i and column j is deleted from matrix A.

The cofactor, C_{ij}, of entry a_{ij} is defined to be $(-1)^{i+j} M_{ij}$ Note: this means that the cofactor is either the same as, or the opposite of the minor, depending on whether $i+j$ is even or odd.

$$A = \begin{bmatrix} 5 & 7 & -1 \\ -2 & 0 & 3 \\ -3 & 1 & 2 \end{bmatrix}$$

A helpful tool for determining whether the sign of the cofactor is the same as or opposite to the sign of the minor. (that is, whether $(-1)^{i+j}$ is positive or negative) is called the Array of Signs: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Now, to find the determinant of matrix A, we expand across any row, or down any column by taking the sum of, the product of, each entry with its cofactor.

$$\begin{vmatrix} 5 & 7 & -1 \\ -2 & 0 & 3 \\ -3 & 1 & 2 \end{vmatrix} = 5\left(\quad \right)\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} + 7\left(\quad \right)\begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix} + (-1)\left(\quad \right)\begin{vmatrix} -2 & 0 \\ -3 & 1 \end{vmatrix}$$

Using a different row,

$$\begin{vmatrix} 5 & 7 & -1 \\ -2 & 0 & 3 \\ -3 & 1 & 2 \end{vmatrix} =$$

Or a column

$$\begin{vmatrix} 5 & 7 & -1 \\ -2 & 0 & 3 \\ -3 & 1 & 2 \end{vmatrix} =$$

Return to 3X3 system

Larger nxn determinants

This method extends to any nxn matrix with the array of signs continuing in the checkerboard pattern.
Note: It is helpful to expand across a row/column with zeros.

$$\begin{vmatrix} 2 & 1 & -3 & 0 \\ -4 & -1 & 0 & 2 \\ 5 & -2 & 3 & 4 \\ 0 & 3 & 1 & 6 \end{vmatrix}$$

````Ans: -494
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